

二元生成自由群的幂单性质

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摘要: 为了研究二元生成自由群的幂单性质, 依据组合群理论, 讨论了本原元素的若干组合方式, 证明了可用于处理幂零的几组新方程. 证明了可以嵌入一般线性群的二元生成自由群是幂单群的充要条件是它的每个本原元素均是幂单的.

关键词: 幂单性; 本原元; 自由群

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The Unipotency of Free Group of Rank Two

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Abstract: To study the unipotency of free group generated by two elements, some combinations of primitive elements are discussed based on the combinatorial group theory, and some new systems benefit to studying the unipotency will be proved. It will prove that the group must be unipotent if and only if it can be imbedded into general linear group $C_{8 \times 8}$ and each primitive element is unipotent.

Key words: unipotency; primitive element; free group

Let $G = \langle A, B \rangle$ be a finitely generated sub-group of $C_{n \times n}$ ($C_{n \times n}$ denotes the general linear group generated by matrix of order n). The group G is called unipotent if every element of G is unipotent, i. e. for each element $P \in G$, there is a positive integer m such that $(P - E)^m = 0$, where E is the unit. Obviously, in this case G is conjugate to a subgroup of the group of all upper-triangular matrices and hence is solvable. What happens if we consider a smaller class of elements of the group? Consider the free group generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. In this case the generators are all unipotent but the group is not unipotent because $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

There is an open question that whether the free group is unipotent when each primitive element is unipotent. Vladimir P. Platonov and Alexander Potapchik^[1] had proved such theorem

Theorem A Let $A, B \in C_{n \times n}$. Assume that every primitive element in A and B is unipotent with Jordan blocks of size at most 3. Then the group generated by A and B is unipotent.

In 2012 O. I. Tavgen^[2] considered the general case and proved

Theorem B The image $\rho(F(x, y)) \subset GL(7, C)$ is unipotent if each primitive element is unipotent.

Until 2013, it is proved that the image $\rho(F(x, y)) \subset GL(n, C)$ is unipotent under the assumptions that each primitive element is unipotent and n is at

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most 7^[5].

The study of unipotency will benefit Lie algebra. It is obvious that a Lie algebra linear-generated by $H, T \in C_{n \times n}$ is nilpotent if and only if the group generated by $A = H + E, B = T + E$ is unipotent.

The case $n = 8$ will be discussed in this paper. Throughout the paper, G denotes the sub-group generated by two elements $A, B \in C_{8 \times 8}$. $H = A - E, T = B - E$, where E is the unit of $C_{8 \times 8}$. $\text{tr} X$ denotes the trail of matrix X . It is easy to prove that $\text{tr} X_{n \times n}^m = n$ for each positive integer m if and only if X is unipotent. $W(P, Q)$ denotes the word in free variables P and Q . We say elements P and Q of G are associative primitive elements if G is generated by P and Q . We also say that primitive element P is associative with Q . Matrix J_n denotes the Jordan-block of order n .

The main result of this paper is

Theorem 1 The free group G generated by two elements $A, B \in C_{8 \times 8}$ is unipotent if and only if each primitive element is unipotent.

To prove this theorem, the following lemmas are useful.

Lemma 1^[5] Let P and Q are associative primitive elements of G . Then the primitive elements which are associative with P must be in the form $P^\alpha Q^\varepsilon P^\beta$, where $\alpha, \beta \in \mathbf{Z}, \varepsilon = \pm 1$.

In the following lemmas we assume that G is a sub-group generated by two elements $A, B \in C_{n \times n}$, and each primitive element of G is unipotent.

Lemma 2 $\text{tr} HB^m = 0, m \in \mathbf{Z}^+$ (1)

$\text{tr} H^m B = 0, m \in \mathbf{Z}^+$ (2)

$\text{tr}(HBHB) = 0$ (3)

$\text{tr}(HBHBHB) = 0$ (4)

Proof It is obvious that AB^m is a primitive element by Lemma 1. So $\text{tr} AB^m = \text{tr} B^m = n$. Thus we obtain (1) because $A = H + E$. If we change AB^m into $A^m B$ and use the induction to m , we can prove (2). Pay attention to primitive elements $A^2 BAB, ABA^2 B$ and AB . We have $\text{tr}(A^2 BAB) = 0, \text{tr}(ABA^2 B) = 0, \text{tr}(ABAB) = 0$ and $\text{tr}(AB)^3 = 0$. From these formulas together with (1) we obtain (3) and (4).

Lemma 3 If $\text{tr} H^i B H^j B = 0$ for all integers i, j with $i + j > k \geq 2$, then

$$\sum_{i=0}^k \frac{\text{tr}(H^i B H^{k-i} B)}{i! (k-i)!} = 0 \quad (5)$$

$$\sum_{i=1}^k \frac{\text{tr}(H^i B H^{k-i} B)}{(i-1)! (k-i-1)!} = 0 \quad (6)$$

Proof Since AB^m is a primitive element by Lemma 1, we obtain $\text{tr} A^m B A^m B = n$. This means that

$$\text{tr} \left(B + mHB + \frac{m(m-1)}{2!} H^2 B + \cdots + \frac{m(m-1) \cdots (m-t+1)}{t!} H^t B \right)^2 = n$$

where $A = H + E$ and $H^{t+1} = 0$ (because A is unipotent).

We may rewrite this formula into the form $\text{tr}(a_0 + a_1 m + \cdots + a_t m^t)^2 = n$ or into the form $F_0 + F_1 m + F_2 m^2 + \cdots + F_{2t} m^{2t} = 0$, where $F_0 = \text{tr}(a_0 a_0) - n, F_1 = \text{tr}(a_0 a_1 + a_1 a_0), \cdots, F_{2t} = \text{tr}(a_t a_t)$. Using Vandermonde determinant, we get $F_k = 0$ for each $k (0 \leq k \leq 2t)$ because m is an arbitrary positive integer. Now we can easily get formula (5) from $F_k = 0$ together with $\text{tr} H^i B H^j B = 0$ for all positive integers $i, j (i + j > k \geq 2)$.

If we change $\text{tr} A^m B A^m B = n$ into $\text{tr} H A^m B H A^m B = 0$, then we can similarly get formula (6). And $\text{tr} H A^m B H A^m B = 0$ can be got from (3) together with the fact that A and $A^m B$ are associative primitive elements.

Lemma 4 If $\text{tr} H^i B H^j B H^k B = 0$ for all integers i, j, k with $i + j + k > m$, then

$$\sum_{i+j+k=m} \frac{\text{tr}(H^i B H^j B H^k B)}{i! j! k!} = 0 \quad (7)$$

$$\sum_{\substack{i+j+k=m \\ i>0, j>0, k>0}} \frac{\text{tr}(H^i B H^j B H^k B)}{(i-1)! (j-1)! (k-1)!} = 0 \quad (8)$$

The proof of Lemma 4 is similar to the proof of Lemma 3. We need only change $\text{tr} A^m B A^m B = n$ into $\text{tr}(A^m B)^3 = n$, change $\text{tr} H A^m B H A^m B = 0$ into $\text{tr}(H A^m B)^3 = 0$. And it is easy to prove that

Lemma 5 If $\text{tr} H^{i_1} B H^{i_2} B \cdots H^{i_k} B = 0$ for all integers $i_1, i_2, \cdots, i_k$ with $i_1 + i_2 + \cdots + i_k > m$ then

$$\sum_{i_1+i_2+\cdots+i_k=m} \frac{\text{tr}(H^{i_1} B H^{i_2} B \cdots H^{i_k} B)}{i_1! i_2! \cdots i_k!} = 0 \text{ Eq}(m, k)$$

Now we see that formulas (1), (2), (5) and (7) are relatively $\text{Eq}(1, m), \text{Eq}(m, 1), \text{Eq}(k, 2)$ and $\text{Eq}(m, 3)$.

Naturally, $\text{eq}(m, k)$ denotes equation

$$\sum_{i_1+i_2+\dots+i_k=m} \frac{\text{tr}(H^{i_1}BH^{i_2}B\cdots H^{i_k}B)}{(i-1)! (i-1)! \cdots (i-1)! k!} = 0 \text{ eq}(m, k)$$

We only proved $\text{eq}(k, 2)$, $\text{eq}(k, 3)$ (which are formulas (6), (8)). Using $\text{tr}[A^m B]^2 A^{m+1} B = n$, $\text{tr}[A^{m+1} B]^2 A^m B = n$, it is induced that

Lemma 6 If $\text{tr}H^iBH^jBH^k B = 0$ for all positive integers i, j, k with $i+j+k > m$, then

$$\sum_{\substack{i+j+k=m \\ i>0}} \frac{\text{tr}(H^iBH^jBH^k B)}{(i-1)! j! k!} = 0 \quad (9)$$

$$\sum_{\substack{i+j+k=m \\ i>0, j>0}} \frac{\text{tr}(H^iBH^jBH^k B)}{(i-1)! (j-1)! k!} = 0 \quad (10)$$

Similarly, we have

Lemma 7 If $\text{tr}H^iBH^jBH^kBH^l B = 0$ for all positive integers i, j, k, l with $i+j+k+l > m$, then

$$\sum_{\substack{i+j+k+l=4m \\ i>0}} \frac{\text{tr}(H^iBH^jBH^kBH^l B)}{(i-1)! j! k! l!} = 0 \quad (11)$$

$$\sum_{\substack{i+j+k+l=4m \\ i>0, k>0}} \frac{\text{tr}(H^iBH^jBH^kBH^l B)}{(i-1)! j! (k-1)! l!} = 0 \quad (12)$$

All the above lemmas can be used in general case. The following lemmas are only used in the case $n=8$. So we assume that G is a free sub-group generated by two elements $A, B \in C_{8 \times 8}$, and each primitive element of G is unipotent.

Lemma 8 If $i+j \geq 5$, then $\text{tr}(H^iBH^jB) = 0$.

Proof We divide the proof into two parts: (I) $H^4 = 0$, (II) $H^4 \neq 0$.

(I) $H^4 = 0$. Since AB^m is a primitive element by Lemma 1, we obtain $\text{tr}A^mBA^mB = 8$. This means that

$$\text{tr}\left(B + mHB + \frac{m(m-1)}{2!}H^2B + \frac{m(m-1)(m-2)}{3!}H^3B\right)^2 = 8$$

where $A = H + E$ and $H^4 = 0$ (because A is unipotent).

We may rewrite this formula into the form $\text{tr}(a_0 + a_1m + a_3m^3)^2 = 8$ or into the form $F_0 + F_1m + F_2m^2 + F_6m^6 = 0$, where $F_0 = \text{tr}(a_0a_0) - n$, $F_1 = \text{tr}(a_0a_1 + a_1a_0)$, $\dots$, $F_6 = \text{tr}(a_3a_3)$. Using Vandermonde determinant, we get $F_k = 0$ for each k ($0 \leq k \leq 6$) because m is an arbitrary positive integer. Paying attention to the fact that $\text{tr}(HBHB) = 0$, we obtain system (13). Similarly, from $\text{tr}A^{m-1}BA^mB = 8$ (by the fact that $A^{m-1}BA^mB$ is primitive element), we obtain system

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{13}{36} & \frac{1}{4} \\ 0 & \frac{1}{3} & 1 & -1 & \frac{5}{6} & -\frac{1}{3} & -\frac{1}{2} \\ 1 & -1 & -1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{9} & \frac{1}{4} \\ -1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} \text{tr}H^2B^2 \\ \text{tr}H^3B^2 \\ \text{tr}HBH^2B \\ \text{tr}HBH^3B \\ \text{tr}H^2BH^3B \\ \text{tr}H^3BH^3B \\ \text{tr}H^2BH^2B \end{pmatrix} = 0 \quad (13)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{13}{12} & \frac{31}{36} & \frac{1}{4} \\ 0 & \frac{1}{3} & 1 & -\frac{5}{3} & \frac{5}{2} & -\frac{17}{12} & -1 \\ 1 & -\frac{3}{2} & -\frac{5}{2} & \frac{8}{3} & \frac{29}{12} & \frac{10}{9} & \frac{5}{4} \\ -2 & \frac{13}{6} & \frac{3}{2} & -\frac{4}{3} & \frac{5}{6} & -\frac{1}{3} & -\frac{1}{2} \end{pmatrix} \times$$

$$\begin{pmatrix} \text{tr}H^2B^2 \\ \text{tr}H^3B^2 \\ \text{tr}HBH^2B \\ \text{tr}HBH^3B \\ \text{tr}H^2BH^3B \\ \text{tr}H^3BH^3B \\ \text{tr}H^2BH^2B \end{pmatrix} = 0 \quad (14)$$

Solving systems (13) and (14) we obtain

$$\text{tr}H^2B^2 = 0, \text{tr}H^3B^2 = 0, \text{tr}HBH^2B = 0, \text{tr}HBH^3B = 0, \text{tr}H^2BH^3B = 0, \text{tr}H^3BH^3B = 0, \text{tr}H^2BH^2B = 0.$$

So we obtain $\text{tr}(H^iBH^jB) = 0$ for all integers i, j .

(II) $H^4 \neq 0$. Let $B = (x_{ij})_{8 \times 8}$. We may divide this part into seven cases by the Jordan-standard form of unipotent matrix A : $\text{diag}(J_5, E)$, $\text{diag}(J_5, J_3)$, diag

(J_5, J_2, E) , $\text{diag}(J_6, E)$, $\text{diag}(J_6, J_2)$, $\text{diag}(J_7, E)$, J_8 .

For example $A = \text{diag}(J_5, E)$. Solving equation $\text{Eq}(4, 1)$, we obtain $x_{51} = 0$. With the help of micro-soft Maple, we have $\text{tr}(H^i B H^j B) = 0$ for all positive integer i, j satisfying $i + j > 6$ by calculating these trails. Solving equations $\text{Eq}(3, 1)$ and $\text{Eq}(6, 2)$, we obtain $x_{41} = x_{52} = 0$. With the help of micro-soft Maple, we can find that $\text{tr}(H^i B H^j B) = 0$ where $i + j \geq 5$.

The proof of other cases is similar.

Lemma 9 For all integers i, j , we have $\text{tr}(H^i B H^j B) = 0$.

Proof Similar to the proof of Lemma 8, divide the proof into two parts: (I) $H^4 = 0$, (II) $H^4 \neq 0$. Then we find that (I) have been proved in Lemma 8. Now let us prove case (II).

(II) $H^4 \neq 0$. It leaves only to prove

$$\begin{aligned} \text{tr} H^4 B^2 &= \text{tr} H^3 B^2 = \text{tr} H^2 B^2 = \text{tr} H B H^2 B = \\ \text{tr} H B H^3 B &= \text{tr} H^2 B H^2 B = 0 \end{aligned}$$

because $\text{tr}(H B H B) = 0$ and $\text{tr}(H^i B H^j B) = 0$ when $i + j \geq 5$.

Once more considering $\text{tr} A^m B A^m B = 8$ and $\text{tr} A^{m-1} B A^m B = 8$, we obtain systems

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{3} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{11}{12} & -1 & 1 & -1 & \frac{2}{3} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{12} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} \end{pmatrix} \times$$

$$\begin{pmatrix} \text{tr} H^4 B^2 \\ \text{tr} H^3 B^2 \\ \text{tr} H^2 B^2 \\ \text{tr} H B H^2 B \\ \text{tr} H B H^3 B \\ \text{tr} H^2 B H^2 B \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{12} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 & -\frac{8}{3} & -1 \\ \frac{23}{12} & -\frac{3}{2} & 1 & -\frac{5}{2} & \frac{8}{3} & \frac{5}{4} \\ -\frac{7}{3} & \frac{13}{6} & -2 & \frac{3}{2} & -\frac{4}{3} & -\frac{1}{2} \end{pmatrix} \times$$

$$\begin{pmatrix} \text{tr} H^4 B^2 \\ \text{tr} H^3 B^2 \\ \text{tr} H^2 B^2 \\ \text{tr} H B H^2 B \\ \text{tr} H B H^3 B \\ \text{tr} H^2 B H^2 B \end{pmatrix} = 0$$

From these systems we can obtain

$$\begin{aligned} \text{tr} H^4 B^2 &= \text{tr} H^3 B^2 = \text{tr} H^2 B^2 = \text{tr} H B H^2 B = \\ \text{tr} H B H^3 B &= \text{tr} H^2 B H^2 B = 0 \end{aligned}$$

Lemma 10 $\text{tr} w(A, B) = 8$ where the whole power of A or B is at most 2 in word $w(A, B)$.

Proof First, we have $\text{tr}(H^i B H^j B) = 0$ for all integers i, j . Similarly $\text{tr}(T^i A T^j A) = 0$ for all integers i, j . This means that $\text{tr} w(H, B) = 0$ where the whole power of B is at most 2, and $\text{tr} w(T, A) = 0$ where the whole power of B is at most 2. So when the whole power of A is at most 2, we have $\text{tr} w(A, B) = \sum \text{tr} w_m(A, T) + \text{tr} A^k = 0 + 8 = 8$ where k is 1 or 2 and there are no more than two A in each word $w_m(A, T)$. So does $\text{tr} w(A, B) = 8$ where the whole power of B is at most 2.

Lemma 11 $\text{tr}(HB)^k = 0$ where $k = 1, 2, \dots, 5$.

Proof We have proved the cases when $k = 1, 2, 3$ (see Lemma 2).

Since AB^2 and AB are associative primitive elements, we have $AB^2 ABAB$ is primitive element by Lemm 1. Consequently,

$$\begin{aligned} 8 &= \text{tr}(AB^2 ABAB) = \text{tr}(HB^2 HBHB) + \\ &\text{tr} B^4 + \sum \text{tr} w_j(H, B) \end{aligned}$$

Where the whole power of H is no more than 2 in each word $w_j(H, B)$. So we have $\text{tr}(HB^2 HBHB) = 0$.

Because AB is primitive element, so we have

$$\begin{aligned} 8 &= \text{tr}(AB)^4 = 4\text{tr}(HB^2 HBHB) + \text{tr} B^4 + \\ &\text{tr}(HB)^4 + \sum \text{tr} w_j(H, B) \end{aligned}$$

Where the whole power of H is no more than 2 in each word $w_j(H, B)$. Consequently, $\text{tr}(HB)^4 = 0$.

Notice that AB and B are associative primitive elements, we obtain $\text{tr}(AB)^3 B^2 = 8$ by Lemma 10. Similarly we obtain

$$\text{tr}(AB)^4 B = \text{tr}(ABAB^2 AB^2) = \text{tr}(AB)^5 = 8$$

Solving these equations and using Lemma 10, we obtain $\text{tr}(HB)^5 = 0$.

Lemma 12 If $\text{tr} H^i B H^j B H^k B H^l B = 0$ for all positive integers i, j, k, l with $i + j + k + l > m$, then

$$\sum_{\substack{i+j+k+l=m \\ i>0, j>0}} \frac{\text{tr}(H^i B H^j B H^k B H^l B)}{(i-1)!(j-1)!k!l!} = 0 \quad (15)$$

$$\sum_{\substack{i+j+k+l=m \\ i>0, j>0, k>0}} \frac{\text{tr}(H^i B H^j B H^k B H^l B)}{(i-1)!(j-1)!(k-1)!l!} = 0 \quad (16)$$

$$\sum_{\substack{i+j+k+l=m \\ i>0, j>0, k>0, l>0}} \frac{\text{tr}(H^i B H^j B H^k B H^l B)}{(i-1)!(j-1)!(k-1)!(l-1)!} = 0 \quad (17)$$

The method of proof has been used above several times, so we ignore the proof. It must be remarked that the results of Lemma 10 has to be used to prove Lemma 12.

Lemma 13 If matrices A and B have a common invariant vector, then group G is unipotent.

This is the direct consequence of theorem B.

Proof of theorem 1

Let $B = (x_{ij})_{8 \times 8}$. We may divide this part into 11 parts by the Jordan-standard form of A as follows: J_8 , $\text{diag}(J_7, E)$, $\text{diag}(J_6, E)$, $\text{diag}(J_6, J_2)$, $\text{diag}(J_5, E)$, $\text{diag}(J_5, J_2, E)$, $\text{diag}(J_5, J_3)$, $\text{diag}(J_7, E)$, $\text{diag}(J_4, J_4)$, $\text{diag}(J_4, J_2, J_2)$, $\text{diag}(J_4, J_3, E)$.

(I) $A = J_8$. Solving equation Eq(7, 1), we obtain $x_{81} = 0$ and $\text{tr} H^i B H^j B = 0$, where $i + j > 12$. Solving Eq(6, 1) and eq(12, 2), we obtain $x_{71} = x_{82} = 0$ and $\text{tr} H^i B H^j B = 0$, where $i + j > 10$. Continue so, solving Eq(m, k) and eq(m, k), we obtain that matrix B has form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} \\ 0 & 0 & x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} \\ 0 & 0 & 0 & x_{54} & x_{55} & x_{56} & x_{57} & x_{58} \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} & x_{67} & x_{68} \\ 0 & 0 & 0 & 0 & 0 & x_{76} & x_{77} & x_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{87} & x_{88} \end{pmatrix} \quad (18)$$

Now we take $M = B^{-1}$, then A and M are associative primitive elements. So M also has form (18). Let $M = (m_{ij})_{8 \times 8}$, then m_{ij} is a function of x_{ij} . Solving system Eq(1, 1), Eq(2, 2), Eq(3, 3), eq(2, 2), eq(3, 3), and (9), (10) (where $m = 3$), together with $m_{81} = m_{82} = m_{83} = 0$, $m_{71} = m_{72} = m_3 = 0$, $m_{61} =$

$m_{62} = m_{63} = 0$, we obtain $x_{ii-1} = 0$ where $i = 1, 2, \dots, 8$.

Thus we proved that A and B are all upper-triangular unipotent matrix. So group G is unipotent.

Similarly, we can prove the parts when matrix A is one of the following forms: $\text{diag}(J_7, E)$, $\text{diag}(J_6, E)$, $\text{diag}(J_6, J_2)$, $\text{diag}(J_5, E)$, $\text{diag}(J_5, J_2, E)$. It must be remarked that the proofs are more complex.

For example, if $A = \text{diag}(J_7, E)$, the proof is divided into two cases (i) $x_{81} = 0$ and (ii) $x_{81} \neq 0$. Not only equations mentioned in the proof of (I) but also equations (11), (12), (15), (16), (17) and $\text{tr}(HB)^k = 0$ (where $k = 1, 2, \dots, 5$) are used. Then we conclude that matrix B has form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & 0 & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} & x_{47} & 0 \\ 0 & 0 & 0 & 0 & x_{55} & x_{56} & x_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{66} & x_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{77} & 0 \\ 1 & x_{82} & x_{83} & x_{84} & x_{85} & x_{86} & x_{87} & x_{88} \end{pmatrix} \text{ or } \begin{pmatrix} x_{11} & * \\ 0 & B_1 \end{pmatrix}.$$

From this, it is induced that A and B have a common invariant vector. So we finish the proof by Lemma 13.

(II) $A = \text{diag}(J_5, J_3)$. Solving equation Eq(4, 1), we obtain $x_{51} = 0$. Now calculating with micro-soft Maple we obtain $\text{tr} H^i B H^j B = 0$ where $i + j > 6$, $\text{tr} H^i B H^j B H^k B = 0$ where $i + j + k > 9$. So we should solve some equations like Eq(3, 1), Eq(6, 2), Eq(9, 3), eq(6, 2), eq(9, 3). In the following part, we ignore the reason and directly write the equations. Solving Eq(3, 1), Eq(6, 2), Eq(9, 3), eq(6, 2), eq(9, 3), we obtain (II-1) $x_{41} = x_{52} = x_{56} = 0$ or (II-2) $x_{41} = x_{52} = x_{81} = 0$. We choose the case (II-1), and the other cases can be similarly proved.

(II-1) $x_{41} = x_{52} = x_{81} = 0$. Solving Eq(3, 1) Eq(5, 2), eq(5, 2), we obtain (II-1-1) $x_{56} = 0$ or (II-1-2) $x_{71} = x_{82} = 0$. As we said early, we choose the case (II-1-1).

(II-1-1) $x_{56} = 0$. Solving Eq(2, 1), Eq(4, 2), Eq(6, 3), Eq(8, 4), eq(4, 2), eq(6, 3),

eq(8, 4) and (11), (12) we obtain three sets of solutions: (II-1-1-1) $x_{31} = x_{42} = x_{46} = x_{53} = x_{57} = x_{86} = 0$, (II-1-1-2) $x_{31} = x_{42} = x_{71} = x_{53} = x_{82} = x_{86} = 0$, (II-1-1-3) $x_{31} = x_{71} = x_{53} = x_{57} = 0$. We only choose the first case.

(II-1-1-1) Solving Eq(3, 2), Eq(4, 3), Eq(6, 4), eq(3, 2), eq(6, 4) and (11), (12), (15), (16), (17), we obtain (i) $x_{71} = x_{82} = 0$, (ii) $x_{36} = x_{47} = x_{82} = x_{87} = 0$, (iii) $x_{36} = x_{47} = x_{58} = 0$

We assert that if P and Q (in G) are associative primitive elements then $\text{tr} P^3 Q^3 = 8$.

First, if P is one of J_8 , $\text{diag}(J_7, E)$, $\text{diag}(J_6, E)$, $\text{diag}(J_6, J_2)$, $\text{diag}(J_5, E)$, $\text{diag}(J_5, J_2, E)$, then, as we proved, G is unipotent. So $\text{tr} P^3 Q^3 = 8$. Second, if P is one of $\text{diag}(J_4, J_4)$, $\text{diag}(J_4, J_2, J_2)$, $\text{diag}(J_4, J_3, E)$, then $(P - E)^4 = 0$. Let $H_p = P - E$, then $P^{-1} = E - H_p + H_p^2 - H_p^3$. Notice, that $P^{-1} Q^3$ is primitive element, we have $\text{tr}(P^{-1} Q^3) = 8$. Considering $\text{tr}(P^{-1} Q^3) = 8$ together with Lemma 10, we have $\text{tr}(H_p^3 Q^3) = 0$. Consequently,

$$8 = \text{tr} Q^3 = \text{tr} Q^3 + \text{tr} H_p^3 Q^3 + 3 \text{tr} H_p^2 Q^3 + 3 \text{tr} H_p Q^3 = \text{tr} P^3 Q^3.$$

Third, if $P = \text{diag}(J_5, J_3)$ we also have $\text{tr}(P^{-1} Q^3) = 8$ where $P^{-1} = E - H_p + H_p^2 - H_p^3 + H_p^4$. Because we have already proved Eq(4, 3), we have $\text{tr}(H_p^4 Q^3) = 0$. There leaves proof similar to the second case. Thus we proved the assertion.

Similarly, we can also prove the fact that $\text{tr}(P^2 Q P Q^2 + P^2 Q^2 P Q) = 16$.

Using Lemma 10, we can reduce the formula:

$$\text{tr}(HB)^6 = \text{tr}(AB - B)^6 = 144 - 6 \text{tr}((AB)^3 B^3 + ABAB^2 AB^3 + ABAB^3 AB^2)$$

where

$$\text{tr}(ABAB^2 AB^3) = \text{tr} [AB^2 (AB)^{-1}]^2 (AB)^2 \cdot (AB^2 (AB)^{-1}) (AB)],$$

$$\text{tr}(ABAB^3 AB^2) = \text{tr} [AB^2 (AB)^{-1}]^2 (AB) \cdot (AB^2 (AB)^{-1}) (AB)^2].$$

Thus we have $\text{tr}(HB)^6 = 0$.

Solving equations Eq(m, k), eq(m, k), $\text{tr}(HB)^k = 0$ (where $k = 1, 2, \dots, 6$), (11), (12), (15), (16), (17) and $\text{tr}(H^k B^3) = 0$ (where $k \in Z^+$), we conclude that B has one of the following forms:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & 0 & x_{33} & x_{34} & x_{35} & 0 & x_{37} & x_{38} \\ 0 & 0 & 0 & x_{44} & x_{45} & 0 & x_{47} & x_{48} \\ 0 & 0 & 0 & 0 & x_{55} & 0 & 0 & x_{58} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} & x_{68} \\ 0 & 0 & x_{73} & x_{74} & x_{75} & 0 & x_{77} & x_{78} \\ 0 & 0 & 0 & x_{84} & x_{85} & 0 & 0 & x_{88} \end{pmatrix},$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & 0 & x_{33} & x_{34} & x_{35} & 0 & x_{37} & x_{38} \\ 0 & 0 & 0 & x_{44} & x_{45} & 0 & 0 & x_{48} \\ 0 & 0 & 0 & 0 & x_{55} & 0 & 0 & x_{58} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} & x_{68} \\ 0 & x_{72} & x_{73} & x_{74} & x_{75} & 0 & x_{77} & x_{78} \\ 0 & 0 & 0 & x_{84} & x_{85} & 0 & 0 & x_{88} \end{pmatrix},$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & 0 & x_{33} & x_{34} & x_{35} & 0 & 0 & x_{38} \\ 0 & 0 & 0 & x_{44} & x_{45} & 0 & 0 & x_{48} \\ 0 & 0 & 0 & 0 & x_{55} & 0 & 0 & x_{58} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} & x_{68} \\ x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & 0 & x_{77} & x_{78} \\ 0 & 0 & 0 & x_{84} & x_{85} & 0 & 0 & x_{88} \end{pmatrix},$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ 0 & x_{32} & x_{33} & x_{34} & x_{35} & 0 & x_{37} & x_{38} \\ 0 & 0 & x_{43} & x_{44} & x_{45} & 0 & 0 & x_{48} \\ 0 & 0 & 0 & 0 & x_{55} & 0 & 0 & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} & x_{68} \\ x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & 0 & x_{77} & x_{78} \\ 0 & 0 & 0 & x_{84} & x_{85} & 0 & 0 & x_{88} \end{pmatrix}.$$

Since B has one of the above forms and $A = \text{diag}(J_5, J_3)$, we conclude that A and B have a common invariant vector. So we finish the proof by Lemma 13.

We ignore the proof of other cases because it is similar.

Remark 1) We proved theorem 1 based on theorem B . But this is not necessary. We can prove step by step the case $n = 1, 2, 3, 4, 5, 6, 7$ with the same

method, and then prove theorem 1.

2) Until we finished the paper, we have not convinced that we can prove the case $n=9$ in this way because of the following reasons. First, we have not proved that all the equations mentioned above are also correct in the case $n=9$. Second, equations are not enough even if we can prove that all the equations mentioned above are also correct. Third, it is difficult to calculate the trails of matrix of order 9 even with the help of Maple.

We deal with only unfinite group because there are diferent methods used in studying finite groups, so theorem 1 can be easily proved when the group is finite^[6-7]. But these the methods are not benefitial to the case of unfinite goups.

Our results are of great benefit to the study of Lie algebra. An interesting problem in studying Lie algebra is nilpotency^[8-9]. Our results can be also rewritten as: The Lie algebra generated by two nilpotent matrices $A, B \in C_{8 \times 8}$ is nilpotent if and only if each primitive element of free group generated by $E+A, E+B$ is unipotent.

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